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ANALYSIS OF HEURISTICS FOR SEQUENCING JOBS ON ONE MACHINE WITH RELEASE DATES AND DELIVERY TIMES

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ABSTRACT

The single machine sequencing problem is considered in which each job has a release date, a processing time and a delivery time. The objective is to find a sequence of jobs which minimises the time by which all jobs are delivered. A heuristic is presented which never deviates by more than 50% from the optimum.

KEY WORDS & PHRASES: *single machine sequencing, release dates, delivery times, heuristics, worst-case performance.*

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1. INTRODUCTION

The usual assumptions about the problem will be adopted. Each of n jobs is to be processed without interruption on a single machine. At any time the machine can process only one job. Job i (i = 1,...,n) is available for processing at time r_i, has a non-zero processing time p_i and has a delivery time q_i. We assume that all r_i, p_i and q_i are integers. The objective is to find a sequence of jobs which minimises the time by which all jobs are delivered.

As stated above the problem is in symmetric form because an equivalent *inverse problem* can be obtained from the original problem by interchanging r_i and q_i for all jobs i. For any constant K, we can define delivery dates for each job i by $d_i = K-q_i$. Minimising the time by which all jobs are delivered in the symmetric form is equivalent to minimising maximum lateness in the modified form.

It has been shown by Lenstra et al. [6] that the problem is NP-hard, which implies that the existence of a polynomial bounded algorithm to solve the problem is unlikely. Implicit enumeration algorithms have been successfully applied to problems with up to 80 jobs by Baker and Su [1], McMahon and Florian [7] and Lageweg et al. [5]. Also Kise et al. [4] have analysed the performance of several heuristics. They showed that each heuristic can deviate by an amount arbitrarily close to 100% from the optimum. In this paper we shall concentrate on the analysis of some heuristics or approximation algorithms. A survey and discussion of some of the research in this area are given by Fisher [2] and Garey et al. [3]. Suppose that T^* denotes the minimum time by which all jobs can be delivered while T_H denotes the time by which all jobs are delivered when the jobs are sequenced using a certain heuristic H. If, whatever the problem data, $T_H \le \rho T^* + \delta$ for specified constants ρ and δ, where ρ is as small as possible, then ρ is called the *worst-case performance ratio* of H.

In Section 2 of this paper the well-known heuristic of Schrage [8] is stated. An analysis indicating the class of problems on which the heuristic performs badly is presented. A modification to the heuristic, suggested by the analysis, is given in Section 3. Its worst-case performance ratio is derived.

2. THE HEURISTIC OF SCHRAGE

Schrage's heuristic can be stated as follows. Suppose that jobs have been sequenced in the first j positions and the machine becomes available at time U_j. Let V_j denote the smallest release date of unsequenced jobs. Then the job to be sequenced in position j+1 is chosen from the set $\{k: k \text{ unsequenced}, r_k \le \max\{U_j, V_j\}\}$ so that its delivery time is as large as possible. If there is a choice, the job with the larger processing time is chosen. It should be noted that this algorithm can also be applied to the inverse problem. The heuristic requires $O(n \log n)$ steps.

We shall refer to Schrage's heuristic as Algorithm S and the corresponding time by which all jobs are delivered as T_S. Kise et al. have shown that $T_S/T^* \le 2 - 3/(P+1)$, where P is the sum of processing times of the jobs. This worst-case performance ratio is achieved for the two-job problem in which $r_1 = 0$, $r_2 = 1$, $p_1 = P-1$, $p_2 = 1$, $q_1 = 0$, $q_2 = P-1$ giving $T_S = 2P-1$ and $T^* = P+1$. Clearly the ratio can be arbitrarily close to 2.

Suppose that the sequence $\sigma = (\sigma(1),\ldots,\sigma(n))$ is generated using Algorithm S and yields

$$T_S = r_{\sigma(i)} + \sum_{h=i}^{j} p_{\sigma(h)} + q_{\sigma(j)}, \tag{1}$$

where $1 \le i \le j \le n$. If there is a choice, it is assumed that i and j are both as small as possible. As i is as small as possible, either job $\sigma(i)$ is the first job in σ or the machine will be idle immediately prior to processing job $\sigma(i)$. In either case we have from the construction of σ that

$$r_{\sigma(i)} \le r_{\sigma(h)} \quad \text{for } h = i,\ldots,j. \tag{2}$$

If $q_{\sigma(j)} \le q_{\sigma(h)}$ for $h = i,\ldots,j$, then it follows from (2) that σ is an optimum sequence. Otherwise we can define a job $\sigma(k)$ such that $i \le k < j$ and such that $q_{\sigma(k)} < q_{\sigma(j)}$ but $q_{\sigma(h)} \ge q_{\sigma(j)}$ for $h = k+1,\ldots,j$. We shall refer to job $\sigma(k)$ as the *interference job* for the sequence σ as it may delay the times at which jobs $\sigma(k+1),\ldots,\sigma(j)$ are delivered by occupying the machine when at least one of the jobs $\sigma(k+1),\ldots,\sigma(j)$ becomes available for processing. Job $\sigma(j)$ is called the *critical job*.

The first result shows that the deviation of T_S from the optimum is bounded by the processing time of the interference job.

THEOREM 1. $T_S - T^* < p_{\sigma(k)}$.

Proof. By considering jobs $\sigma(k+1),\ldots,\sigma(j)$ the following lower bound for T^* is obtained:

$$T^* \ge \min\{r_{\sigma(k+1)},\ldots,r_{\sigma(j)}\} + \sum_{h=k+1}^{j} p_{\sigma(h)} + q_{\sigma(j)}.$$

Subtracting from (1) yields

$$T_S - T^* \le r_{\sigma(i)} + \sum_{h=i}^{k} p_{\sigma(h)} - \min\{r_{\sigma(k+1)},\ldots,r_{\sigma(j)}\}. \tag{3}$$

Now during the application of Algorithm S, jobs $\sigma(k+1),\ldots,\sigma(j)$ were not available for processing at time $r_{\sigma(i)} + \sum_{h=i}^{k-1} p_{\sigma(h)}$. Equivalently $r_{\sigma(i)} + \sum_{h=i}^{k-1} p_{\sigma(h)} < \min\{r_{\sigma(k+1)},\ldots,r_{\sigma(j)}\}$. Substituting in (3) we obtain

$$T_S - T^* < p_{\sigma(k)},$$

which completes the proof. □

Now if P denotes the sum of the processing times, then clearly $T^* \geq P$. The corollary below follows immediately from Theorem 1. It will be applied in the next section.

COROLLARY 1. $T_S/T^* < 1 + p_{\sigma(k)}/P$.

The next result shows that the deviation of T_S from the optimum is also bounded by the delivery time of the critical job.

THEOREM 2. $T_S - T^* \leq q_{\sigma(j)}$.

Proof. We have the following lower bound for T^*:

$$T^* \geq \min\{r_{\sigma(i)},\ldots,r_{\sigma(j)}\} + \sum_{h=i}^{j} p_{\sigma(h)} + \min\{q_{\sigma(i)},\ldots,q_{\sigma(j)}\}.$$

Applying (2) and using the non-negativity of delivery times we obtain

$$T^* \geq r_{\sigma(i)} + \sum_{h=i}^{j} p_{\sigma(h)}.$$

Subtracting from (1) yields

$$T_S - T^* \leq q_{\sigma(j)},$$

which is the required result. □

Suppose Algorithm S is applied to the inverse problem to give a sequence $\bar{\sigma}$

with a corresponding time $T_{\bar{S}}$ by which all jobs are delivered. Let $\bar{\sigma}(\bar{j})$ denote the critical job for the sequence $\bar{\sigma}$. Then by symmetry we have the following result.

COROLLARY 2. $T_{\bar{S}}-T^{*} \leq r_{\bar{\sigma}(\bar{j})}$.

From Theorem 2 and its Corollary, it would seem likely that improved results could be obtained by applying Algorithm S to the inverse problem whenever $\max\{r_1,\ldots,r_n\} < \max\{q_1,\ldots,q_n\}$. This confirms the results of the empirical study by Kise et al. Also it might be expected that the branch and bound algorithm of McMahon and Florian, which is based on applying Algorithm S at each node of the search tree, should be applied to the inverse problem whenever $\max\{r_1,\ldots,r_n\} < \max\{q_1,\ldots,q_n\}$. This is in accordance with the computational results of Lageweg et al.

It is possible to use Theorem 2 to obtain an approximate estimate of the expected performance of Algorithm S. We shall assume, as is usual for empirical testing, that delivery times are integers from a uniform distribution $[1,2\mu_q-1]$, where $2\mu_q$ is integer. Then the expected value of the maximum delivery time is bounded above by $2\mu_q-1$. Thus from Theorem 2 we have

$$E(T_S-T^{*}) \leq 2\mu_q-1.$$

Furthermore if μ_p is the mean processing time, then $E(T^{*}) \geq n\mu_p$, giving

$$E(T_S)/E(T^{*}) \leq 1 + (2\mu_q-1)/(n\mu_p). \qquad (4)$$

This bound was derived by first finding the maximum deviation and then taking expectations. It is likely that a better bound could be obtained by taking expectations at an earlier stage of the analysis. However, it is satisfying to realise that $E(T_S)/E(T^*) \rightarrow 1$ as $n \rightarrow \infty$.

3. THE MODIFIED HEURISTIC

It was seen from Theorem 1 that the maximum deviation of T_S from T^* is bounded by the processing time of the interference job. Only if this processing time is large do we have a large deviation. The basis of the modified heuristic is to apply Schrage's algorithm successively, each time constraining the interference job to be processed after the critical job in the following sequence. Eventually any single job with a large processing time will cease to be an interference job. The procedure is stated more formally below.

A sequence of times by which all jobs are delivered $T_S^{(0)},\ldots,T_S^{(n-1)}$ with corresponding sequences $\sigma^{(0)},\ldots,\sigma^{(n-1)}$ is generated, where $T_S^{(0)}$ and $\sigma^{(0)}$ are obtained by applying Algorithm S to the original problem. If $\sigma^{(t)}(k^{(t)})$ and $\sigma^{(t)}(j^{(t)})$ are the interference job and the critical job for the sequence $\sigma^{(t)}$, then the constraint that $\sigma^{(t)}(j^{(t)})$ precedes $\sigma^{(t)}(k^{(t)})$ in sequence $\sigma^{(t+1)}$ is added. If $j = \sigma^{(t)}(j^{(t)})$ and $k = \sigma^{(t)}(k^{(t)})$, this constraint is easily implemented by setting $r_k = r_j$. When $t = n-1$ or when no interference job exists the procedure is terminated. Otherwise Algorithm S is reapplied to obtain $T_S^{(t+1)}$ and $\sigma^{(t+1)}$. Thus, a maximum time $T_{S'}$ $= \min\{T_S^{(0)},\ldots,T_S^{(n-1)}\}$ by which all jobs are delivered is obtained, where $T_S^{(h)} = T_S^{(t)}$ for $h = t+1,\ldots,n-1$ when termination occurs directly after the computation of $T_S^{(t)}$. This modified algorithm, which will be called Algorithm S', requires $O(n^2 \log n)$ steps. Use of the result that $(\sigma^{(t+1)}(1),\ldots,\sigma^{(t+1)}(k^{(t)}-1)) = (\sigma^{(t)}(1),\ldots,\sigma^{(t)}(k^{(t)}-1))$ will reduce computation.

The remainder of this paper will be devoted to proving that Algorithm S' has a worst-case performance ratio of 3/2. Firstly, the following lemma shows that $T_{S'}/T^* < 3/2$ if, at some stage, a constraint is added to the problem which changes the minimum time by which all jobs are delivered.

LEMMA 1. *Either there exists an optimum sequence to the original problem satisfying all added constraints or* $\min\{T_S^{(0)},\ldots,T_S^{(n-2)}\}/T^* < 3/2$.

Proof. Suppose that there exists no optimum sequence to the original problem which satisfies all added constraints. Then we can identify the first constraint that changes the minimum time by which all jobs are delivered. Let this constraint be derived from the sequence $\sigma^{(t)}$, where $0 \le t \le n-2$. To simplify the notation we write σ instead of $\sigma^{(t)}$. Then suppose

$$T_S^{(t)} = r_{\sigma(i)} + \sum_{h=i}^{j} p_{\sigma(h)} + q_{\sigma(j)}, \tag{5}$$

where $\sigma(j)$ is the critical job and let $\sigma(k)$ be the interference job. Now if $p_{\sigma(k)} \le P/2$, where P denotes the sum of the processing times, then $T_S^{(t)}/T^* < 3/2$ from Corollary 1.

It remains to show that if $p_{\sigma(k)} > P/2$, then $T_S^{(t)}/T^* < 3/2$ in the case that there exists an optimum sequence to the original problem in which $\sigma(k)$ precedes $\sigma(j)$. We have here that

$$T^* \ge r_{\sigma(k)} + p_{\sigma(k)} + p_{\sigma(j)} + q_{\sigma(j)}. \tag{6}$$

Subtracting (6) from (5) yields

$$T_S^{(t)} - T^* \le r_{\sigma(i)} - r_{\sigma(k)} + \sum_{h=i}^{k-1} p_{\sigma(h)} + \sum_{h=k+1}^{j-1} p_{\sigma(h)}.$$

Now by applying (2) we have $r_{\sigma(i)} \le r_{\sigma(k)}$. Therefore

$$T_S^{(t)} - T^* \le P - p_{\sigma(k)} < P/2,$$

which implies that $T_S^{(t)}/T^* < 3/2$ since $P \leq T^*$. This completes the proof. □

The main result will now be proved.

THEOREM 3. $T_{S'}/T^* < 3/2$.

Proof. Again let P denote the sum of all processing times. If the processing times of all jobs are less than or equal to P/2, then the result follows from Corollary 1. Otherwise there exists a single job u such that $p_u > P/2$. If, at some stage t, job u is not the interference job, then the result follows from Corollary 1. Otherwise job u is the interference job at each stage. If there is no optimum sequence to the original problem which satisfies all the added constraints, then the result follows from Lemma 1. In all other cases after t constraints have been added, where $t \leq n-1$, a sequence will be generated in which job u is sequenced last. For this sequence the interference job cannot be job u. Thus the processing time of the interference job is less than P/2 and the result follows from Corollary 1. This completes the proof. □

We now show that the bound of Theorem 3 is the best possible for this heuristic. Consider the 3-job problem with $r_1 = 0$, $r_2 = 1$, $r_3 = (P+1)/2$, $p_1 = (P-1)/2$, $p_2 = (P-1)/2$, $p_3 = 1$, $q_1 = 0$, $q_2 = (P-3)/2$, $q_3 = (P-1)/2$, where P is an odd integer and $P \geq 3$. Algorithm S yields $\sigma^{(0)} = (1,2,3)$ and $T_S^{(0)} = (3P-1)/2$. Job 2 is the interference job and job 3 is the critical job. The constraint that job 3 precedes job 2 in $\sigma^{(1)}$ is added by setting $r_2 = r_3 = (P+1)/2$. Reapplying Algorithm S yields $\sigma^{(1)} = (1,3,2)$ with $T_S^{(1)} = (3P-1)/2$. At this stage there is no interference job, so the algorithm terminates with $T_{S'} = (3P-1)/2$. Clearly (2,3,1) is the optimum sequence with T^* = P+1. Thus $T_{S'}/T^* = 3/2 - 2/(P+1)$, which can be arbitrarily close to 3/2.

To summarise, we have shown that Schrage's heuristic can be modified to ensure that a solution within 50% of the optimum is always produced. This represents an improvement on the most commonly used heuristics that were analysed by Kise et al., which could all deviate by an amount arbitrarily close to 100% from the optimum.

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